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**THE PROCEDURAL EGALITARIAN SOLUTION AND
EGALITARIAN STABLE GAMES**

By

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The procedural egalitarian solution and egalitarian stable games

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Abstract

This paper studies the procedural egalitarian solution on the class of egalitarian stable games. By deriving several axiomatic characterizations involving consistency and monotonicity, we show that the procedural egalitarian solution satisfies various desirable properties and unites many egalitarian concepts defined in the literature. Moreover, we illustrate the computational implications of these characterizations and relate the class of egalitarian stable games to other well-known classes.

Keywords: egalitarianism; transferable utility games; procedural egalitarian solution; egalitarian stability

JEL classification: C71

1 Introduction

Egalitarianism is a social and economic principle pursuing the notion of equality. This principle stems from the belief that all humans are fundamentally equal and should be treated equally. Such a point of view is often justified using philosophical thought experiments in which members of a society negotiate about social goals behind the veil of ignorance, i.e. without being aware of their identity, characteristics, and natural abilities and endowments a priori.

However, in many interactive situations, egalitarianism is not the only desirable value that plays a role. Agents often distinguish themselves in terms of contributions, rights, needs, power, or responsibility in such a way that a purely equal treatment is not necessarily considered fair. In these cases, society looks for a trade-off between egalitarianism and other fairness principles.

In this paper, we interpret egalitarianism as a principle for distributive justice, i.e. the nature of socially just allocations of goods and bads. In this context, a widely used measure for egalitarianism in economic distributions such as incomes, wealth, and taxes is the so-called Lorenz criterion. Roughly speaking, a certain reward or cost allocation Lorenz dominates another allocation if the former cumulatively assigns to each subgroup of ex post richest agents less than the latter does.

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In a seminal paper, Dutta and Ray (1989) applied the Lorenz criterion in the context of coalitional games, where the focus is on allocating joint revenues among cooperating players. Here, strict equal division is not really satisfactory since it does not take into account the economic possibilities of subcoalitions. Instead, motivated by a coherent use of egalitarian norms, Dutta and Ray (1989) applied the Lorenz criterion to the self-introduced Lorenz core. Remarkably, and particularly caused by the Lorenz core not being closed, this results in at most one payoff allocation, despite the partial ordering generated by the Lorenz criterion. However, the proposed solution lacks existence for many games in its domain. In fact, a general existence result has not been provided up to the present day, but existence is guaranteed for the special class of convex games. There, the Dutta and Ray (1989) solution belongs to the core and Lorenz dominates every other core allocation.

Inspired by the aforementioned work, several authors continued the line of research aiming for an appropriate trade-off between egalitarianism and coalitional rationality in the context of transferable utility games. Dutta and Ray (1991) applied the Lorenz criterion to the equal division core. For this solution, existence is guaranteed under the mild condition that the underlying game is cohesive, a weak form of superadditivity and balancedness, which substantially widens the potential domain of applications. However, the Dutta and Ray (1991) solution does not possess the extremely appealing uniqueness property of the Dutta and Ray (1989) solution. The same holds for the approach taken by Hougaard, Peleg, and Thorlund-Petersen (2001) and Arin and Iñarra (2001), where the Lorenz criterion is applied to the core. This extends the Dutta and Ray (1989) solution for convex games to the class of balanced games, but generally results in a set of multiple payoff allocations. Recently, Dietzenbacher, Borm, and Hendrickx (2017) introduced the procedural egalitarian solution for which existence and uniqueness is guaranteed for any transferable utility game. Moreover, it coincides with the Dutta and Ray (1989) solution on the class of convex games.

Dutta (1990) axiomatically characterized the Dutta and Ray (1989) solution on the class of convex games. Klijn, Slikker, Tijs, and Zarzuelo (2000) presented several reformulations of these characterizations. In both works, a significant role is played by the reduced game property of Davis and Maschler (1965), to which we refer as max-consistency. In the earlier development of the theory of coalitional games, max-consistency has already been exploited for axiomatizations of the prenucleolus (cf. Sobolev (1975)), the prekernel (cf. Peleg (1986)), and the core (cf. Peleg (1986)). Later, in an egalitarian context, it was not only part of axiomatizations of the Dutta and Ray (1989) solution, but also of an analogous result for the egalitarian core (cf. Arin and Iñarra (2001)).

This paper initiates an axiomatic study of the procedural egalitarian solution. The procedural egalitarian solution is based on the result of an iterative procedure in which intercoalitional egalitarian considerations are central. This procedure converges to a steady state where each player has acquired a certain egalitarian claim which is attainable in at least one egalitarian admissible coalition. Taking these claims into account, the procedural egalitarian solution allocates the worth of the grand coalition in an egalitarian way among the cooperating players.

We focus on the large class of egalitarian stable games where the grand coalition is egalitarian admissible. There, the procedural egalitarian solution turns out to be a Lorenz undominated element of the core and satisfies various desirable properties. We start from the original result of Dutta (1990) where the Dutta and Ray (1989) solution for convex games is characterized by an alternative solution for two-player games, called constrained egalitarianism, and max-consistency. These axioms do not comprise a unique solution on the larger class of egalitarian stable games. However, an elementary characterization of

constrained egalitarianism does lead to a full axiomatization of the procedural egalitarian solution. Interestingly, this axiomatization induces an alternative computational method for the underlying iterative procedure.

A second characterization tells that the procedural egalitarian solution is the unique selector from the egalitarian core which satisfies aggregate monotonicity. In this respect, the procedural egalitarian solution can also be interpreted as a trade-off between egalitarian coalitional rationality and aggregate monotonicity. Arin, Kuipers, and Vermeulen (2008) provided sufficient conditions for egalitarianism based on symmetry and core contraction independence. This inspires us to derive a third axiomatization of the procedural egalitarian solution using coalitional rationality, aggregate monotonicity, symmetry, and core contraction independence.

In the analysis, an explicit comparison is made with the equal division solution, which simply prescribes equal division of the worth of the grand coalition, and the coalitional Nash solution (cf. Compte and Jehiel (2010)), which assigns to any balanced game the core element with maximal payoff product. Besides, we observe that the procedural egalitarian solution unites many egalitarian concepts defined in the literature.

We conclude by relating the class of egalitarian stable games with other well-known subclasses of balanced games which contain all convex games. In particular, we show that all large core games (cf. Sharkey (1982)) are egalitarian stable. Several open questions could serve as fruitful suggestions for future research.

This paper is organized in the following way. Section 2 formally describes the procedural egalitarian solution based on Dietzenbacher et al. (2017) and derives some useful elementary results. Section 3 formulates several axiomatic characterizations, discusses the computational implications, and presents connections with other egalitarian concepts. Section 4 elaborates more on the class of egalitarian stable games.

2 The procedural egalitarian solution

Let \mathcal{N} be a nonempty and finite set. A *transferable utility game* is a pair (N, v) in which $N \subseteq \mathcal{N}$ is a nonempty set of *players* and $v : 2^N \rightarrow \mathbb{R}_+$ assigns to each *coalition* $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}_+$ such that $v(\emptyset) = 0$.¹ Let $\text{TU}^{\mathcal{N}}$ denote the class of all transferable utility games.

Let $(N, v) \in \text{TU}^{\mathcal{N}}$. The *imputation set* is defined by

$$I(N, v) = \left\{ x \in \mathbb{R}^N \left| \sum_{i \in N} x_i = v(N), \forall i \in N : x_i \geq v(\{i\}) \right. \right\}$$

and the *core* is defined by

$$C(N, v) = \left\{ x \in \mathbb{R}^N \left| \sum_{i \in N} x_i = v(N), \forall S \in 2^N : \sum_{i \in S} x_i \geq v(S) \right. \right\}.$$

The game (N, v) is *imputation admissible* if $I(N, v) \neq \emptyset$, *balanced* if $C(N, v) \neq \emptyset$, and *convex* (cf. Shapley (1971)) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \in 2^N$. Convex games are balanced, and balanced games are imputation admissible.

¹Although completely unnecessary for the procedural egalitarian solution, we restrict to nonnegative games in order to make a solid comparison with the coalitional Nash solution. However, none of the formal results depends on this restriction.

A *solution* on $\text{TU}_*^N \subseteq \text{TU}^N$ is a function f which assigns to any $(N, v) \in \text{TU}_*^N$ a payoff allocation $f(N, v) \in \mathbb{R}^N$ for which $\sum_{i \in N} f_i(N, v) = v(N)$. Throughout this paper, f is the generic notation for a solution.

We focus on the *procedural egalitarian solution* introduced by Dietzenbacher et al. (2017). This solution is based on the result of an iterative procedure in which intercoalitional egalitarian considerations are central. The formal definition of this procedure follows after an illustrative example.

Example 1

Let $(N, v) \in \text{TU}^N$ be the game with $N = \{1, 2, 3\}$ for which the worth of each coalition and the egalitarian procedure underlying the procedural egalitarian solution are presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	5	0	0	8	0	0	9
$\chi^{v,1}(S)$	(5, ·, ·)	(·, 0, ·)	(·, ·, 0)	(4, 4, ·)	(0, ·, 0)	(·, 0, 0)	(3, 3, 3)
$\chi^{v,2}(S)$	(5, ·, ·)	(·, 0, ·)	(·, ·, 0)	(5, 3, ·)	(5, ·, -5)	(·, 0, 0)	(5, 2, 2)
$\chi^{v,3}(S)$	(5, ·, ·)	(·, 3, ·)	(·, ·, 0)	(5, 3, ·)	(5, ·, -5)	(·, 3, -3)	(5, 3, 1)
$\chi^{v,k}(S) (k \geq 4)$	(5, ·, ·)	(·, 3, ·)	(·, ·, 1)	(5, 3, ·)	(5, ·, 1)	(·, 3, 1)	(5, 3, 1)

The function $\chi^{v,1}$ divides the worth of each coalition equally among its members. Players can fix their allocated payoff in a coalition if none of its members is allocated a higher payoff in any other coalition. This is the case for player 1 with a payoff of 5 in coalition $\{1\}$. In the next iteration, $\chi^{v,2}$ assigns in each coalition to player 1 the payoff of 5 and divides the remaining worth equally among the other members. Again, players can fix their allocated payoff in a coalition if none of its members is allocated a higher payoff in any other coalition. This is still the case for player 1 with a payoff of 5 in coalition $\{1\}$, but this is now also the case for player 2 with a payoff of 3 in coalition $\{1, 2\}$. This procedure continues in this way and guarantees that all players fix a payoff at some point. \triangle

The egalitarian procedure

Let $(N, v) \in \text{TU}^N$. Define $P^{v,0} = \emptyset$. Let $k \in \mathbb{N}$. The function $\chi^{v,k}$ assigns to each $S \in 2^N \setminus \{\emptyset\}$ the payoff allocation $\chi^{v,k}(S) \in \mathbb{R}^S$ defined by

$$\chi_i^{v,k}(S) = \begin{cases} \gamma_i^{v,k-1} & \text{if } i \in S \cap P^{v,k-1}; \\ \frac{v(S) - \sum_{j \in S \cap P^{v,k-1}} \gamma_j^{v,k-1}}{|S \setminus P^{v,k-1}|} & \text{if } i \in S \setminus P^{v,k-1}. \end{cases}$$

The collection $\mathcal{A}^{v,k} \subseteq 2^N \setminus \{\emptyset\}$ is defined by

$$\mathcal{A}^{v,k} = \left\{ S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \chi_i^{v,k}(S) = v(S), \forall i \in S \forall T \in 2^N : i \in T : \chi_i^{v,k}(T) \leq \chi_i^{v,k}(S) \right\}.$$

The set $P^{v,k} \in 2^N \setminus \{\emptyset\}$ is defined by $P^{v,k} = \bigcup_{S \in \mathcal{A}^{v,k}} S$. The vector $\gamma^{v,k} \in \mathbb{R}^{P^{v,k}}$ is defined by $\gamma_i^{v,k} = \chi_i^{v,k}(S)$ for all $i \in P^{v,k}$, where $S \in \mathcal{A}^{v,k}$ and $i \in S$. \circ

This procedure is well-defined and guarantees that all players fix a payoff within a number of iterations which is bounded by the number of players in the underlying game, i.e. for any $(N, v) \in \text{TU}^N$, we have $P^{v,k} \subseteq P^{v,k+1}$ for all $k \in \mathbb{N}$ and $P^{v,k} = N$ for some $k \leq |N|$. A useful observation is that this potential payoff does not increase over the iterations. As long as some players have not fixed a payoff yet, their allocated payoffs in a next iteration will be at most their allocated payoffs in the current iteration. This is formally described by the following lemma.

Lemma 2.1

Let $(N, v) \in \text{TU}^N$ and let $k \in \mathbb{N}$. Then $\chi_i^{v, k+1}(S) \leq \chi_i^{v, k}(S)$ for all $i \in N \setminus P^{v, k}$ and each $S \in 2^N$ with $i \in S$.

Proof. Let $i \in N \setminus P^{v, k}$ and let $S \in 2^N$ be such that $i \in S$. Then

$$\begin{aligned}
\chi_i^{v, k+1}(S) &= \frac{v(S) - \sum_{j \in S \cap P^{v, k}} \gamma_j^{v, k}}{|S \setminus P^{v, k}|} \\
&= \frac{v(S) - \sum_{j \in S \cap P^{v, k-1}} \gamma_j^{v, k} - \sum_{j \in S \cap (P^{v, k} \setminus P^{v, k-1})} \gamma_j^{v, k}}{|S \setminus P^{v, k}|} \\
&\leq \frac{v(S) - \sum_{j \in S \cap P^{v, k-1}} \gamma_j^{v, k-1} - \sum_{j \in S \cap (P^{v, k} \setminus P^{v, k-1})} \chi_j^{v, k}(S)}{|S \setminus P^{v, k}|} \\
&= \frac{|S \setminus P^{v, k-1}| \chi_i^{v, k}(S) - |S \cap (P^{v, k} \setminus P^{v, k-1})| \chi_i^{v, k}(S)}{|S \setminus P^{v, k}|} \\
&= \frac{|S \setminus P^{v, k}| \chi_i^{v, k}(S)}{|S \setminus P^{v, k}|} \\
&= \chi_i^{v, k}(S).
\end{aligned}$$

□

This also means that, when a player fixes a payoff for the first time, this payoff is the lowest among all members of the corresponding coalitions. In general, it implies that for each player there exists a coalition where all members can attain their fixed payoffs and all these payoffs are at least as large as the fixed payoff of this particular player. This remark will be of great importance for the derivation of several properties of the procedural egalitarian solution. The procedural egalitarian solution takes the fixed payoffs from the procedure and the coalitions in which they are attainable into account to prescribe a payoff allocation for the worth of the grand coalition. The fixed payoffs are called the *egalitarian claims* of the players and coalitions in which they are attainable are called *egalitarian admissible*.

The procedural egalitarian solution

Let $(N, v) \in \text{TU}^N$. The iteration $n^v \leq |N|$ is defined by $n^v = \min\{k \in \mathbb{N} \mid P^{v, k} = N\}$. The vector of *egalitarian claims* $\hat{\gamma}^v \in \mathbb{R}^N$ is defined by $\hat{\gamma}^v = \gamma^{v, n^v}$. The collection of *egalitarian admissible coalitions* $\hat{\mathcal{A}}^v \subseteq 2^N \setminus \{\emptyset\}$ is defined by $\hat{\mathcal{A}}^v = \mathcal{A}^{v, n^v}$. The set of *strong egalitarian claimants* $D^v \in 2^N$ is defined by $D^v = \bigcap \{S \in \hat{\mathcal{A}}^v \mid \forall_{T \in \hat{\mathcal{A}}^v} : S \not\subseteq T\}$. The *procedural egalitarian solution* $\text{PES}(N, v) \in \mathbb{R}^N$ is defined by

$$\text{PES}(N, v) = \left((\hat{\gamma}_i^v)_{i \in D^v}, (\min\{\hat{\gamma}_i^v, \lambda\})_{i \in N \setminus D^v} \right),$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \text{PES}_i(N, v) = v(N)$. ○

Example 2

Let $(N, v) \in \text{TU}^N$ be the game from Example 1. Then $n^v = 3$, $\hat{\gamma}^v = (5, 3, 1)$, $\hat{\mathcal{A}}^v = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$, and $D^v = N$. Hence, $\text{PES}(N, v) = (5, 3, 1)$. △

Let $(N, v) \in \text{TU}^N$. The vector of egalitarian claims is an *aspiration* (cf. Bennett (1983)), i.e. $\sum_{j \in S} \hat{\gamma}_j^v \geq v(S)$ for all $S \in 2^N$, and for each $i \in N$ there exists an $S \in 2^N$ with $i \in S$ for which $\sum_{j \in S} \hat{\gamma}_j^v = v(S)$. Lemma 2.1 even implies that for each $i \in N$ there exists an $S \in 2^N$ with $i \in S$ for which $\sum_{j \in S} \hat{\gamma}_j^v = v(S)$ and $\hat{\gamma}_i^v \leq \hat{\gamma}_j^v$ for all $j \in S$. The collection of egalitarian admissible coalitions consists of those coalitions in which all members can attain their egalitarian claims, i.e.

$$\hat{\mathcal{A}}^v = \left\{ S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \hat{\gamma}_i^v = v(S) \right\}.$$

The result of the egalitarian procedure is interpreted as a claims problem in which the worth of the grand coalition is the endowment and the players are entitled to their egalitarian claims. Members of all inclusion-wise maximal egalitarian admissible coalitions are called *strong egalitarian claimants*. The procedural egalitarian solution assigns to the strong claimants their claims and divides the remaining worth of the grand coalition as equal as possible among the other players under the condition that players are not allocated more than their claims.

Example 3

Let $(N, v) \in \text{TU}^N$ be the game with $N = \{1, 2, 3\}$ for which the worth of each coalition and the first iteration of the egalitarian procedure are presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	0	0	1	1	0	1
$\chi^{v,1}(S)$	$(\bar{0}, \cdot, \cdot)$	$(\cdot, \bar{0}, \cdot)$	$(\cdot, \cdot, \bar{0})$	$(\frac{1}{2}, \frac{1}{2}, \cdot)$	$(\frac{1}{2}, \cdot, \frac{1}{2})$	$(\cdot, \bar{0}, \bar{0})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Then $n^v = 1$, $\hat{\gamma}^v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $\hat{\mathcal{A}}^v = \{\{1, 2\}, \{1, 3\}\}$, and $D^v = \{1\}$. Hence, $\text{PES}(N, v) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.² \triangle

Note that the grand coalition is not egalitarian admissible in Example 3. By contrast, the grand coalition is egalitarian admissible in Example 1 and Example 2. In that case, the vector of egalitarian claims is a feasible aspiration, all players are strong claimants, and the procedural egalitarian solution simply assigns to all players their claims. Such games are called *egalitarian stable*.

Egalitarian stability A game $(N, v) \in \text{TU}^N$ is *egalitarian stable* if $N \in \hat{\mathcal{A}}^v$. \circ

In fact, a game is egalitarian stable if and only if the procedural egalitarian solution is an element of its core. Let TU_{es}^N denote the class of all egalitarian stable games. Dietzenbacher et al. (2017) showed that convex games are egalitarian stable, and that egalitarian stable games are balanced. For two-player games, egalitarian stability is equivalent to convexity and balancedness. On the class of convex games, the procedural egalitarian solution coincides with the Dutta and Ray (1989) solution, i.e. it is the Lorenz undominated element of the Lorenz core and the Lorenz undominated element of the core.

²Surprisingly, the procedural egalitarian solution coincides with the Dutta and Ray (1991) solution, i.e. it is the Lorenz undominated element of the equal division core. The Dutta and Ray (1989) solution does not exist, i.e. there is no Lorenz undominated element of the Lorenz core.

3 Egalitarianism and coalitional rationality

This section studies the procedural egalitarian solution on the class of egalitarian stable games. There, the procedural egalitarian solution is an element of the core, which implies that it is an element of the imputation set. In other words, it satisfies individual rationality and coalitional rationality.

Individual rationality $f(N, v) \in I(N, v)$ for all $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. ○

Coalitional rationality $f(N, v) \in C(N, v)$ for all $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. ○

On the class of convex games, a strict subclass of egalitarian stable games, the procedural egalitarian solution coincides with the Dutta and Ray (1989) solution. Dutta (1990) showed that the Dutta and Ray (1989) solution is the unique solution on the class of convex games satisfying constrained egalitarianism and max-consistency. Constrained egalitarianism is an alternative solution for games with two players in which the worth of the grand coalition is divided as equal as possible subject to individual rationality.³

Constrained egalitarianism

$$f_i(N, v) = \begin{cases} \max\{v(\{i\}), \frac{1}{2}v(N)\} & \text{if } v(\{i\}) \geq v(N \setminus \{i\}); \\ v(N) - v(N \setminus \{i\}) & \text{if } v(\{i\}) \leq v(N \setminus \{i\}). \end{cases}$$

for all $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ with $|N| = 2$ and each $i \in N$. ○

On the class of games with two players, constrained egalitarianism is equivalent to individual rationality and the equal division upper bound. The equal division upper bound requires that no player is allocated more than the maximal average worth in the game. This could be desirable from an egalitarian point of view.

The equal division upper bound

$$f_i(N, v) \leq \max_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$$

for all $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ and each $i \in N$. ○

Lemma 3.1

If a solution satisfies individual rationality and the equal division upper bound, then it satisfies constrained egalitarianism.

Proof. Let f be a solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$. Assume that f satisfies individual rationality and the equal division upper bound. Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ with $|N| = 2$. Denote $N = \{1, 2\}$ such that $v(\{1\}) \geq v(\{2\})$. Then $v(\{1\}) + v(\{2\}) \leq v(N)$ and $f_1(N, v) + f_2(N, v) = v(N)$.

Suppose that $v(\{1\}) \leq \frac{1}{2}v(N)$. By the equal division upper bound, $f_1(N, v) \leq \frac{1}{2}v(N)$ and $f_2(N, v) \leq \frac{1}{2}v(N)$. This means that $f_1(N, v) = f_2(N, v) = \frac{1}{2}v(N)$.

Now suppose that $v(\{1\}) \geq \frac{1}{2}v(N)$. By individual rationality, $f_1(N, v) \geq v(\{1\})$. By the equal division upper bound, $f_1(N, v) \leq v(\{1\})$. This means that $f_1(N, v) = v(\{1\})$ and $f_2(N, v) = v(N) - f_1(N, v)$. Hence, f satisfies constrained egalitarianism. □

³The standard solution for $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ with $|N| = 2$ is for each $i \in N$ defined by

$$f_i(N, v) = \frac{1}{2} (v(N) + v(\{i\}) - v(N \setminus \{i\})).$$

The other way around, a solution satisfies individual rationality and the equal division upper bound on the class of two-player games if it satisfies constrained egalitarianism. Since the Dutta and Ray (1989) solution for convex games satisfies individual rationality and the equal division upper bound, the result of Dutta (1990) implies that the Dutta and Ray (1989) solution is the unique solution on the class of convex games satisfying individual rationality, the equal division upper bound, and max-consistency.

Max-consistency is based on reduced games as introduced by Davis and Maschler (1965) in line with the following thought experiment. Suppose that some players reevaluate their assigned payoffs within a specific subgroup. Serving as an appropriate benchmark, a new cooperative game on this subgroup is defined in which the economic possibilities of all its members are reflected. In this so-called reduced game, each coalition is allowed to cooperate with any combination of players outside the subgroup, provided that they receive their assigned payoffs. The corresponding solution is max-consistent if it prescribes for this reduced game the same payoffs as for the original game.

Max-consistency $f_T(N, v) = f(T, v_T^f)$ for all $(N, v) \in \text{TU}_{\text{es}}^N$ and each $T \in 2^N \setminus \{\emptyset\}$, where

$$v_T^f(S) = \begin{cases} v(N) - \sum_{i \in N \setminus T} f_i(N, v) & \text{if } S = T; \\ \max_{R \subseteq N \setminus T} \{v(S \cup R) - \sum_{i \in R} f_i(N, v)\} & \text{if } S \in 2^T \setminus \{\emptyset, T\}; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

○

The procedural egalitarian solution satisfies individual rationality on the class of egalitarian stable games. As derived in the Appendix, it also satisfies the equal division upper bound and max-consistency.⁴ In particular, this means that the procedural egalitarian solution satisfies constrained egalitarianism and max-consistency, which indirectly proves that the procedural egalitarian solution indeed coincides with the Dutta and Ray (1989) solution on the class of convex games.

However, the procedural egalitarian solution is not the unique solution on the class of egalitarian stable games satisfying constrained egalitarianism and max-consistency. Compte and Jehiel (2010) introduced the *coalitional Nash solution* CN, which assigns to any game $(N, v) \in \text{TU}_{\text{es}}^N$ the payoff allocation

$$\text{CN}(N, v) = \operatorname{argmax}_{x \in C(N, v)} \prod_{i \in N_+^v} x_i,$$

where $N_+^v = \{i \in N \mid \exists_{x \in C(N, v)} : x_i > 0\}$. The coalitional Nash solution also satisfies constrained egalitarianism and max-consistency, but is in general different from the procedural egalitarian solution. This is illustrated in the following example.

Example 4

Let $(N, v) \in \text{TU}_{\text{es}}^N$ be the game with $N = \{1, 2, 3, 4\}$ for which the worth of each coalition is given by

$$v(S) = \begin{cases} 8 & \text{if } S = N; \\ 6 & \text{if } S \in \{\{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}; \\ 5 & \text{if } S \in \{\{2, 3\}, \{1, 2, 3\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

⁴In fact, the procedural egalitarian solution satisfies the equal division upper bound on the full class of transferable utility games.

The procedural egalitarian solution is given by $\text{PES}(N, v) = (0, 2, 3, 3)$. The corresponding payoff product equals zero, whereas there exist several core elements with a positive payoff product, e.g. $(1, 1, 4, 2)$. This means that the procedural egalitarian solution does not coincide with the coalitional Nash solution. It does not coincide with the Dutta and Ray (1989) solution DR either, which is given by $\text{DR}(N, v) = (1, 1, 3, 3)$. Note that the Dutta and Ray (1989) solution does not belong to the core. Moreover, the convex reduced game $(\{1, 2\}, v_{\{1,2\}}^{\text{DR}}) \in \text{TU}_{\text{es}}^N$ is given by

$$v_{\{1,2\}}^{\text{DR}}(S) = \begin{cases} 2 & \text{if } S \in \{\{2\}, \{1, 2\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

By constrained egalitarianism, $\text{DR}(\{1, 2\}, v_{\{1,2\}}^{\text{DR}}) = (0, 2)$. This means that the Dutta and Ray (1989) solution does not satisfy max-consistency in general. \triangle

The purpose of Example 4 is twofold. First, it shows that the Dutta and Ray (1989) solution does not satisfy max-consistency on its full domain of existence. Second, it shows that the procedural egalitarian solution, the coalitional Nash solution, and the Dutta and Ray (1989) solution are all generally different. There is no unique solution for egalitarian stable games satisfying constrained egalitarianism and max-consistency. However, there is a unique solution satisfying individual rationality, the equal division upper bound, and max-consistency. It is the procedural egalitarian solution.

Theorem 3.2

The procedural egalitarian solution is the unique solution on TU_{es}^N satisfying individual rationality, the equal division upper bound, and max-consistency.

Proof. The procedural egalitarian solution satisfies individual rationality. By Lemma A.1, the procedural egalitarian solution satisfies the equal division upper bound. By Lemma A.2, the procedural egalitarian solution satisfies max-consistency. Let f be a solution on TU_{es}^N satisfying individual rationality, the equal division upper bound, and max-consistency.

First, we show that f satisfies coalitional rationality. Suppose that f does not satisfy coalitional rationality. Let $(N, v) \in \text{TU}_{\text{es}}^N$ and let $S \in 2^N$ be such that $\sum_{i \in S} f_i(N, v) < v(S)$. By individual rationality, $1 < |S| < |N|$. Let $i \in S$. Then

$$\begin{aligned} f_i(N, v) &= f_i\left((N \setminus S) \cup \{i\}, v_{(N \setminus S) \cup \{i\}}^f\right) \geq v_{(N \setminus S) \cup \{i\}}^f(\{i\}) \\ &= \max_{R \subseteq S \setminus \{i\}} \left\{ v(R \cup \{i\}) - \sum_{i \in R} f_i(N, v) \right\} \geq v(S) - \sum_{i \in S \setminus \{i\}} f_i(N, v) > f_i(N, v), \end{aligned}$$

where the first equality follows from max-consistency and the first inequality follows from individual rationality. This is a contradiction. Hence, f satisfies coalitional rationality.

Next, we show by induction on the number of players that f is uniquely defined. For all $(N, v) \in \text{TU}_{\text{es}}^N$ with $|N| = 1$, we have $f(N, v) = v(N)$. Let $k \in \mathbb{N}$ and assume that $f(N, v)$ is uniquely defined for all $(N, v) \in \text{TU}_{\text{es}}^N$ with $|N| \leq k$. Let $(N, v) \in \text{TU}_{\text{es}}^N$ with $|N| = k + 1$. Let $S \in 2^N \setminus \{\emptyset\}$ be such that $\frac{v(S)}{|S|} \geq \frac{v(T)}{|T|}$ for all $T \in 2^N \setminus \{\emptyset\}$. By the equal division upper bound, $f_i(N, v) \leq \frac{v(S)}{|S|}$ for all $i \in S$. By coalitional rationality, $\sum_{i \in S} f_i(N, v) \geq v(S)$. This means that $f_i(N, v) = \frac{v(S)}{|S|}$ for all $i \in S$. By max-consistency, $f_{N \setminus S}(N, v) = f(N \setminus S, v_{N \setminus S}^f)$, where $f(N \setminus S, v_{N \setminus S}^f)$ is uniquely defined since $|N \setminus S| \leq k$. This means that $f(N, v)$ is uniquely defined. Hence, $f(N, v) = \text{PES}(N, v)$. \square

Theorem 3.2 has interesting computational implications for the procedural egalitarian solution. In accordance with the proof of the uniqueness part, the procedural egalitarian solution can also be obtained by iteratively assigning the maximal average worth to all members of the corresponding coalitions and subsequently considering the reduced game with respect to the other players. This exactly corresponds with the computation method for the (max-)reduced equal split-off set introduced by Llerena and Mauri (2016). Besides, it is in turn equivalent to a third computation method in which the maximal average remaining worth is iteratively assigned to all members of the corresponding coalitions. This is illustrated in the following example.

Example 5

Let $(N, v) \in \text{TU}_{\text{es}}^N$ be the game with $N = \{1, 2, 3, 4\}$ for which the worth of each coalition is given by

$$v(S) = \begin{cases} 24 & \text{if } S = N; \\ 18 & \text{if } S = \{2, 3\}; \\ 10 & \text{if } S = \{3\}; \\ 6 & \text{if } S = \{1\}; \\ 0 & \text{otherwise.} \end{cases}$$

The first three iterations of the egalitarian procedure are partially presented in the following table.

S	$\{1\}$	$\{3\}$	$\{2, 3\}$	$\{1, 2, 3, 4\}$
$v(S)$	6	10	18	24
$\chi^{v,1}(S)$	(<u>6</u> , ·, ·, ·)	(·, ·, <u>10</u> , ·)	(·, 9, 9, ·)	(<u>6</u> , <u>6</u> , <u>6</u> , <u>6</u>)
$\chi^{v,2}(S)$	(<u>6</u> , ·, ·, ·)	(·, ·, <u>10</u> , ·)	(·, <u>8</u> , <u>10</u> , ·)	(<u>6</u> , 4, <u>10</u> , 4)
$\chi^{v,3}(S)$	(<u>6</u> , ·, ·, ·)	(·, ·, <u>10</u> , ·)	(·, <u>8</u> , <u>10</u> , ·)	(<u>6</u> , <u>8</u> , <u>10</u> , <u>0</u>)

The procedural egalitarian solution is given by $\text{PES}(N, v) = (6, 8, 10, 0)$. Note that players do not acquire their egalitarian claims in order.⁵ Alternatively, the procedural egalitarian solution can be computed by iteratively assigning the maximal average remaining worth to all members of the corresponding coalitions. This is presented in the following table.

S	$\{1\}$	$\{3\}$	$\{2, 3\}$	$\{1, 2, 3, 4\}$
$v(S)$	6	10	18	24
	(<u>6</u> , ·, ·, ·)	(·, ·, <u>10</u> , ·)	(·, 9, 9, ·)	(<u>6</u> , <u>6</u> , <u>6</u> , <u>6</u>)
	(<u>6</u> , ·, ·, ·)	(·, ·, <u>10</u> , ·)	(·, <u>8</u> , <u>10</u> , ·)	($4\frac{2}{3}$, $4\frac{2}{3}$, <u>10</u> , $4\frac{2}{3}$)
	(<u>6</u> , ·, ·, ·)	(·, ·, <u>10</u> , ·)	(·, <u>8</u> , <u>10</u> , ·)	(3, <u>8</u> , <u>10</u> , 3)
	(<u>6</u> , ·, ·, ·)	(·, ·, <u>10</u> , ·)	(·, <u>8</u> , <u>10</u> , ·)	(<u>6</u> , <u>8</u> , <u>10</u> , <u>0</u>)

△

Let $(N, v) \in \text{TU}_{\text{es}}^N$. For any payoff allocation $x \in \mathbb{R}^N$, the *maximum surplus* (cf. Davis and Maschler (1965)) of player $i \in N$ over $j \in N \setminus \{i\}$ is defined by

$$s_{ij}^x(N, v) = \max_{S \in 2^N : i \in S, j \notin S} \left\{ v(S) - \sum_{h \in S} x_h \right\}.$$

The *egalitarian core* (cf. Arin and Iñarra (2001)) is defined by

$$EC(N, v) = \{x \in C(N, v) \mid \forall i, j \in N : x_i > x_j : s_{ij}^x(N, v) = 0\}.$$

⁵However, in convex games, players do acquire their egalitarian claims from high to low.

Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. For any payoff allocation $x \in \mathbb{R}^N$, let $\bar{x} \in \mathbb{R}^{|N|}$ be obtained from x by permuting its coordinates in such a way that $\bar{x}_1 \geq \dots \geq \bar{x}_{|N|}$. The payoff allocation $y \in \mathbb{R}^N$ *Lorenz dominates* $x \in \mathbb{R}^N$, denoted by $y \prec_{\text{Lor}} x$, if $\bar{y} \neq \bar{x}$ and $\sum_{i=1}^k \bar{y}_i \leq \sum_{i=1}^k \bar{x}_i$ for all $k \in \{1, \dots, |N|\}$. The *strong egalitarian core* (cf. Hougaard et al. (2001) and Arin and Iñarra (2001))⁶ is defined by

$$\text{SEC}(N, v) = \{x \in C(N, v) \mid \forall y \in C(N, v) : y \not\prec_{\text{Lor}} x\}.$$

In other words, an allocation is an element of the egalitarian core if and only if no other core allocation can be obtained by an equalizing bilateral transfer (cf. Arin et al. (2008)). Similarly, an allocation is an element of the strong egalitarian core if and only if no other core allocation can be obtained by a finite number of equalizing bilateral transfers. Indeed, the strong egalitarian core is a subset of the egalitarian core. A solution satisfies egalitarian coalitional rationality if it is an element of the egalitarian core, and satisfies strong egalitarian coalitional rationality if it is an element of the strong egalitarian core.

Egalitarian coalitional rationality $f(N, v) \in EC(N, v)$ for all $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. ○

Strong egalitarian coalitional rationality $f(N, v) \in \text{SEC}(N, v)$ for all $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. ○

On the class of egalitarian stable games, the procedural egalitarian solution satisfies strong egalitarian coalitional rationality, i.e. it is a Lorenz undominated element of the core. This is another indirect proof that the procedural egalitarian solution coincides with the Dutta and Ray (1989) solution on the class of convex games.

Lemma 3.3

The procedural egalitarian solution satisfies strong egalitarian coalitional rationality.

Proof. Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. Denote $x = \text{PES}(N, v)$. Let $y \in C(N, v)$. We show that $y = x$ if $\sum_{i=1}^k \bar{y}_i \leq \sum_{i=1}^k \bar{x}_i$ for all $k \in \{1, \dots, |N|\}$. Assume that $\sum_{i=1}^k \bar{y}_i \leq \sum_{i=1}^k \bar{x}_i$ for all $k \in \{1, \dots, |N|\}$. Define $R_0 = \emptyset$ and $R_k = \{i \in N \mid \forall j \in N \setminus R_{k-1} : x_j \leq x_i\}$ for all $k \in \mathbb{N}$. Then $R_{k-1} \subseteq R_k$ for all $k \in \mathbb{N}$ and $R_k = N$ for all $k \geq |N|$. We show by induction that $y_{R_k} = x_{R_k}$ for all $k \in \mathbb{N}$. Clearly, $y_i = x_i$ for all $i \in R_0$.

Let $k \in \mathbb{N}$ and assume that $y_i = x_i$ for all $i \in R_{k-1}$. Then $y_i \leq x_i$ for all $i \in R_k$. Let $i \in R_k \setminus R_{k-1}$ and let $S \in \hat{\mathcal{A}}^v$ with $i \in S$ be such that $x_i \leq x_j$ for all $j \in S$. Then $S \subseteq R_k$ and

$$v(S) \leq \sum_{j \in S} y_j \leq \sum_{j \in S} x_j = v(S).$$

This means that $y_j = x_j$ for all $j \in S$. In general, $y_i = x_i$ for all $i \in R_k$. Hence, the procedural egalitarian solution satisfies strong egalitarian coalitional rationality. □

Corollary 3.4

The procedural egalitarian solution is the unique solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$ satisfying strong egalitarian coalitional rationality, the equal division upper bound, and max-consistency.

⁶Arin and Iñarra (2001) called this the *Lorenz stable set*.

The solution f on $\text{TU}_{\text{es}}^{\mathcal{N}}$ for which $f(N, v) = (4, 3, 3, 2, 7)$ if $N = \{1, 2, 3, 4, 5\}$ and

$$v(S) = \begin{cases} 19 & \text{if } S = N; \\ 12 & \text{if } S = \{1, 2, 3, 4\}; \\ 7 & \text{if } S \in \{\{5\}, \{1, 2\}, \{1, 3\}\}; \\ 0 & \text{otherwise,} \end{cases}$$

and $f(N, v) = \text{PES}(N, v)$ otherwise, satisfies strong egalitarian coalitional rationality and the equal division upper bound, but does not satisfy max-consistency. The coalitional Nash solution satisfies strong egalitarian coalitional rationality and max-consistency, but does not satisfy the equal division upper bound. The *equal division solution* ED, which assigns to any game $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ the payoff allocation

$$\text{ED}(N, v) = \left(\frac{v(N)}{|N|} \right)_{i \in N},$$

satisfies the equal division upper bound and max-consistency, but does not satisfy individual rationality. This means that the properties in Theorem 3.2 are independent and remain independent when individual rationality is strengthened to strong egalitarian coalitional rationality as in Corollary 3.4.

A consequence of violating the equal division upper bound is that the coalitional Nash solution does not satisfy monotonicity properties. If the worth of the grand coalition turns out to be larger than expected, some players can be worse off. This property, to which we refer as aggregate monotonicity, was first described by Megiddo (1974). Young (1985) introduced the coalitional monotonicity property which requires that no member can be worse off when the worth of a coalition increases, *ceteris paribus*. Clearly, coalitional monotonicity implies aggregate monotonicity.

Coalitional monotonicity $f_S(N, v) \leq f_S(N, v')$ for all $(N, v), (N, v') \in \text{TU}_{\text{es}}^{\mathcal{N}}$ and each $S \in 2^N$ for which $v(S) \leq v'(S)$ and $v(T) = v'(T)$ for all $T \in 2^N \setminus \{S\}$. \circ

Aggregate monotonicity $f(N, v) \leq f(N, v')$ for all $(N, v), (N, v') \in \text{TU}_{\text{es}}^{\mathcal{N}}$ for which $v(N) \leq v'(N)$ and $v(S) = v'(S)$ for all $S \subset N$. \circ

Hokari (2000) showed that the Dutta and Ray (1989) solution satisfies coalitional monotonicity on the class of convex games. Young (1985) showed that coalitional rationality and coalitional monotonicity are not compatible on the class of balanced games. The following example shows that, on the class of balanced games, egalitarian coalitional rationality and aggregate monotonicity are not compatible either.

Example 6

Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ be the game from Example 3. Then $EC(N, v) = \{(1, 0, 0)\}$. Let $(N, v') \in \text{TU}_{\text{es}}^{\mathcal{N}}$ be the game with $N = \{1, 2, 3\}$ for which the worth of each coalition is presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v'(S)$	0	0	0	1	1	0	$1\frac{1}{2}$

Then $v(N) \leq v'(N)$ and $v(S) = v'(S)$ for all $S \subset N$. However, $EC(N, v') = \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. This means that egalitarian coalitional rationality and aggregate monotonicity are not compatible. \triangle

However, on the class of egalitarian stable games, all these properties are compatible, as they are satisfied by the procedural egalitarian solution.⁷ Moreover, egalitarian coalitional rationality and aggregate monotonicity even characterize the procedural egalitarian solution.

Theorem 3.5

The procedural egalitarian solution is the unique solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$ satisfying egalitarian coalitional rationality and aggregate monotonicity.

Proof. By Lemma 3.3, the procedural egalitarian solution satisfies egalitarian coalitional rationality. By Lemma A.3, the procedural egalitarian solution satisfies aggregate monotonicity. Let f be a solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$ satisfying egalitarian coalitional rationality and aggregate monotonicity. Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. We show that $f(N, v)$ is uniquely defined. Define $Q_0 = \emptyset$. Let $k \in \mathbb{N}$. Define

$$\alpha_k = \max_{S \in 2^N : S \not\subseteq Q_{k-1}} \left\{ \frac{v(S) - \sum_{j \in S \cap Q_{k-1}} f_j(N, v)}{|S \setminus Q_{k-1}|} \right\}$$

$$\text{and } Q_k = \bigcup_{S \in 2^N : S \not\subseteq Q_{k-1}} \left\{ \frac{v(S) - \sum_{j \in S \cap Q_{k-1}} f_j(N, v)}{|S \setminus Q_{k-1}|} \right\} \cup Q_{k-1}.$$

We show by induction that for all $k \in \mathbb{N}$, $f_i(N, v) \leq \alpha_k$ for each $i \in N \setminus Q_{k-1}$, and $f_i(N, v) = \alpha_k$ for each $i \in Q_k \setminus Q_{k-1}$.

Suppose that $f_i(N, v) > \alpha_1$ for some $i \in N$. Define $R_1 = \arg\max_{i \in N} f_i(N, v)$. Then $R_1 \neq N$. Define $\beta_1 \in \mathbb{R}$ such that

$$\max_{i \in N} f_i(N, v) > \beta_1 > \max \left\{ \alpha_1, \max_{i \in N \setminus R_1} f_i(N, v) \right\}.$$

Define $(N, v'_1) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ by

$$v'_1(S) = \begin{cases} \sum_{i \in R_1} f_i(N, v) + |N \setminus R_1| \beta_1 & \text{if } S = N; \\ v(S) & \text{otherwise.} \end{cases}$$

Then $v'_1(N) > v(N)$. By aggregate monotonicity, $f(N, v'_1) \geq f(N, v)$. Suppose that $f_i(N, v'_1) > f_i(N, v)$ for some particular $i \in N$. Then

$$\sum_{j \in S} f_j(N, v'_1) > \sum_{j \in S} f_j(N, v) \geq v(S) = v'_1(S)$$

for all $S \subset N$ with $i \in S$. This means that $s_{ij}^{f(N, v'_1)}(N, v'_1) < 0$ for all $j \in N \setminus \{i\}$. Then egalitarian coalitional rationality implies that $f_i(N, v'_1) \leq f_j(N, v'_1)$ for all $j \in N$. In other words, $f_i(N, v'_1) = f_i(N, v)$ for each $i \in R_1$, and $f_i(N, v'_1) = \beta_1$ for each $i \in N \setminus R_1$. Then

$$\sum_{i \in S} f_i(N, v'_1) > \sum_{i \in S} \alpha_1 \geq \sum_{i \in S} \frac{v(S)}{|S|} = |S| \frac{v(S)}{|S|} = v(S) = v'_1(S)$$

for all $S \in 2^N \setminus \{\emptyset, N\}$. In particular, this means that $s_{ij}^{f(N, v'_1)}(N, v'_1) < 0$ for all $i \in R_1$ and $j \in N \setminus R_1$. This contradicts that f satisfies egalitarian coalitional rationality. Hence, $f_i(N, v) \leq \alpha_1$ for each $i \in N$.

⁷In fact, the procedural egalitarian solution satisfies aggregate monotonicity on the full class of transferable utility games. In this respect, the procedural egalitarian solution can also be interpreted as a trade-off between egalitarian coalitional rationality and aggregate monotonicity.

By coalitional rationality,

$$\sum_{i \in S} f_i(N, v) \geq v(S) = |S|\alpha_1 = \sum_{i \in S} \alpha_1 \geq \sum_{i \in S} f_i(N, v)$$

for all $S \in 2^N \setminus \{\emptyset\}$ for which $\frac{v(S)}{|S|} = \alpha_1$. This means that $f_i(N, v) = \alpha_1$ for each $i \in Q_1$.

Let $k \in \mathbb{N}$ and assume that for all $h \in \{1, \dots, k\}$, $f_i(N, v) \leq \alpha_h$ for each $i \in N \setminus Q_{h-1}$, and $f_i(N, v) = \alpha_h$ for each $i \in Q_h \setminus Q_{h-1}$. Suppose that $f_i(N, v) > \alpha_{k+1}$ for some $i \in N \setminus Q_k$. Define $R_{k+1} = \operatorname{argmax}_{i \in N \setminus Q_k} f_i(N, v)$. Then $R_{k+1} \neq N \setminus Q_k$. Define $\beta_{k+1} \in \mathbb{R}$ such that

$$\alpha_k \geq \max_{i \in N \setminus Q_k} f_i(N, v) > \beta_{k+1} > \max \left\{ \alpha_{k+1}, \max_{i \in N \setminus (Q_k \cup R_{k+1})} f_i(N, v) \right\}.$$

Define $(N, v'_{k+1}) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ by

$$v'_{k+1}(S) = \begin{cases} \sum_{i \in Q_k \cup R_{k+1}} f_i(N, v) + |N \setminus (Q_k \cup R_{k+1})|\beta_{k+1} & \text{if } S = N; \\ v(S) & \text{otherwise.} \end{cases}$$

Then $v'_{k+1}(N) > v(N)$. By aggregate monotonicity, $f(N, v'_{k+1}) \geq f(N, v)$. Suppose that $f_i(N, v'_{k+1}) > f_i(N, v)$ for some particular $i \in N$. Then

$$\sum_{j \in S} f_j(N, v'_{k+1}) > \sum_{j \in S} f_j(N, v) \geq v(S) = v'_{k+1}(S)$$

for all $S \subset N$ with $i \in S$. This means that $s_{ij}^{f(N, v'_{k+1})}(N, v'_{k+1}) < 0$ for all $j \in N \setminus \{i\}$. Then egalitarian coalitional rationality implies that $f_i(N, v'_{k+1}) \leq f_j(N, v'_{k+1})$ for all $j \in N$. In other words, $f_i(N, v'_{k+1}) = f_i(N, v)$ for each $i \in Q_k \cup R_{k+1}$, and $f_i(N, v'_{k+1}) = \beta_{k+1}$ for each $i \in N \setminus (Q_k \cup R_{k+1})$. Then

$$\begin{aligned} \sum_{i \in S} f_i(N, v'_{k+1}) &= \sum_{i \in S \cap Q_k} f_i(N, v'_{k+1}) + \sum_{i \in S \setminus Q_k} f_i(N, v'_{k+1}) \\ &> \sum_{i \in S \cap Q_k} f_i(N, v) + \sum_{i \in S \setminus Q_k} \alpha_{k+1} \\ &\geq \sum_{i \in S \cap Q_k} f_i(N, v) + \sum_{i \in S \setminus Q_k} \frac{v(S) - \sum_{j \in S \cap Q_k} f_j(N, v)}{|S \setminus Q_k|} \\ &= \sum_{i \in S \cap Q_k} f_i(N, v) + |S \setminus Q_k| \frac{v(S) - \sum_{j \in S \cap Q_k} f_j(N, v)}{|S \setminus Q_k|} \\ &= \sum_{i \in S \cap Q_k} f_i(N, v) + v(S) - \sum_{j \in S \cap Q_k} f_j(N, v) \\ &= v(S) \\ &= v'_{k+1}(S) \end{aligned}$$

for all $S \subset N$ for which $S \not\subseteq Q_k$. In particular, this means that $s_{ij}^{f(N, v'_{k+1})}(N, v'_{k+1}) < 0$ for all $i \in R_{k+1}$ and $j \in N \setminus (Q_k \cup R_{k+1})$. This contradicts that f satisfies egalitarian coalitional rationality. Hence, $f_i(N, v) \leq \alpha_{k+1}$ for each $i \in N \setminus Q_k$.

By coalitional rationality,

$$\begin{aligned}
\sum_{i \in S \setminus Q_k} f_i(N, v) &= \sum_{i \in S} f_i(N, v) - \sum_{i \in S \cap Q_k} f_i(N, v) \\
&\geq v(S) - \sum_{i \in S \cap Q_k} f_i(N, v) \\
&= |S \setminus Q_k| \alpha_{k+1} \\
&= \sum_{i \in S \setminus Q_k} \alpha_{k+1} \\
&\geq \sum_{i \in S \setminus Q_k} f_i(N, v)
\end{aligned}$$

for all $S \in 2^N$ with $S \not\subseteq Q_k$ for which $\frac{v(S) - \sum_{j \in S \cap Q_k} f_j(N, v)}{|S \setminus Q_k|} = \alpha_{k+1}$. This means that $f_i(N, v) = \alpha_{k+1}$ for each $i \in Q_{k+1} \setminus Q_k$. \square

Corollary 3.6

The procedural egalitarian solution is the unique solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$ satisfying strong egalitarian coalitional rationality and coalitional monotonicity.

The coalitional Nash solution satisfies strong egalitarian coalitional rationality, but does not satisfy aggregate monotonicity. The equal division solution satisfies coalitional monotonicity, but does not satisfy egalitarian coalitional rationality. This means that the properties in Theorem 3.5 are independent and remain independent when egalitarian coalitional rationality is strengthened to strong egalitarian coalitional rationality, and aggregate monotonicity is strengthened to coalitional monotonicity, as in Corollary 3.6.

Arin et al. (2008) provided sufficient conditions for egalitarian core allocations using the properties symmetry and core contraction independence.⁸ Suppose that the core of a given game turns out to be smaller than expected, but the original solution is still an element of this smaller core. Then the core contraction independence property states that the solution should not change. This can be motivated by the veil of ignorance argument. If members of a society agree upon a certain core allocation without being aware of their own role, then it is plausible that the selected allocation will not be modified when the set of alternatives shrinks.

Dietzenbacher et al. (2017) showed that the procedural egalitarian solution satisfies symmetry. We show that the procedural egalitarian solution also satisfies core contraction independence and derive a third axiomatic characterization in terms of coalitional rationality, aggregate monotonicity, and these two properties.

Symmetry $f_i(N, v) = f_j(N, v)$ for all $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ and each $i, j \in N$ for which $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. \circ

Core contraction independence $f(N, v) = f(N, v')$ for all $(N, v), (N, v') \in \text{TU}_{\text{es}}^{\mathcal{N}}$ for which $f(N, v) \in C(N, v')$ and $C(N, v') \subseteq C(N, v)$. \circ

⁸Arin et al. (2008) called this latter property *independence of irrelevant core allocations*.

Theorem 3.7

The procedural egalitarian solution is the unique solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$ satisfying coalitional rationality, aggregate monotonicity, symmetry, and core contraction independence.

Proof. The procedural egalitarian solution satisfies coalitional rationality and symmetry. By Lemma A.3, the procedural egalitarian solution satisfies aggregate monotonicity. By Lemma A.4, the procedural egalitarian solution satisfies core contraction independence. Let f be a solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$ satisfying coalitional rationality, aggregate monotonicity, symmetry, and core contraction independence. Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. We show that $f(N, v)$ is uniquely defined. Define $Q_0 = \emptyset$ and $\beta = \min_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. Let $k \in \mathbb{N}$. Define

$$\alpha_k = \max_{S \in 2^N : S \not\subseteq Q_{k-1}} \left\{ \frac{v(S) - \sum_{j \in S \cap Q_{k-1}} f_j(N, v)}{|S \setminus Q_{k-1}|} \right\}$$

$$\text{and } Q_k = \bigcup_{S \in 2^N : S \not\subseteq Q_{k-1}} \left\{ \frac{v(S) - \sum_{j \in S \cap Q_{k-1}} f_j(N, v)}{|S \setminus Q_{k-1}|} \right\} \cup Q_{k-1}.$$

We show by induction that for all $k \in \mathbb{N}$, $f_i(N, w) = \alpha_k$ for each $i \in Q_k \setminus Q_{k-1}$ and any $(N, w) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ for which

$$\begin{aligned} v(N) \leq w(S) &\leq \sum_{i \in Q_{k-1}} f_i(N, v) + |N \setminus Q_{k-1}| \alpha_k && \text{if } S = N; \\ w(S) &= v(S) && \text{if } S \subseteq Q_k; \\ |S| \beta &\leq w(S) \leq v(S) && \text{otherwise.} \end{aligned}$$

Define $(N, v'_1) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ by

$$v'_1(S) = \begin{cases} |N| \alpha_1 & \text{if } S = N; \\ |S| \beta & \text{otherwise.} \end{cases}$$

Then $v'_1(N) \geq v(N)$ and $v'_1(S) \leq v(S)$ for all $S \subset N$. By symmetry, $f_i(N, v'_1) = \alpha_1$ for all $i \in N$. Let $(N, w) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ be such that

$$\begin{aligned} v(N) \leq w(S) &\leq |N| \alpha_1 && \text{if } S = N; \\ w(S) &= v(S) && \text{if } S \subseteq Q_1; \\ |S| \beta &\leq w(S) \leq v(S) && \text{otherwise.} \end{aligned}$$

Then $w(N) \leq v'_1(N)$ and $v'_1(S) \leq w(S) \leq \sum_{i \in S} f_i(N, v'_1)$ for all $S \subset N$. By core contraction independence and aggregate monotonicity, $f_i(N, w) \leq \alpha_1$ for all $i \in N$. By coalitional rationality,

$$\sum_{i \in S} f_i(N, w) \geq w(S) = v(S) = |S| \alpha_1 = \sum_{i \in S} \alpha_1 \geq \sum_{i \in S} f_i(N, w)$$

for all $S \in 2^N \setminus \{\emptyset\}$ for which $\frac{v(S)}{|S|} = \alpha_1$. This means that $f_i(N, w) = \alpha_1$ for each $i \in Q_1$.

Let $k \in \mathbb{N}$ and assume that for all $h \in \{1, \dots, k\}$, $f_i(N, w) = \alpha_h$ for each $i \in Q_h \setminus Q_{h-1}$ and any $(N, w) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ for which

$$\begin{aligned} v(N) \leq w(S) &\leq \sum_{i \in Q_{h-1}} f_i(N, v) + |N \setminus Q_{h-1}| \alpha_h \text{ if } S = N; \\ w(S) &= v(S) \text{ if } S \subseteq Q_h; \\ |S| \beta &\leq w(S) \leq v(S) \text{ otherwise.} \end{aligned}$$

Define $(N, v'_{k+1}) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ by

$$v'_{k+1}(S) = \begin{cases} \sum_{i \in Q_k} f_i(N, v) + |N \setminus Q_k| \alpha_{k+1} & \text{if } S = N; \\ v(S) & \text{if } S \subseteq Q_k; \\ |S| \beta & \text{otherwise.} \end{cases}$$

Then $f_i(N, v'_{k+1}) = \alpha_h$ for all $i \in Q_h \setminus Q_{h-1}$ and any $h \in \{1, \dots, k\}$. In other words, $f_i(N, v'_{k+1}) = f_i(N, v)$ for all $i \in Q_k$. By symmetry, this means that $f_i(N, v'_{k+1}) = \alpha_{k+1}$ for all $i \in N \setminus Q_k$. Let $(N, w) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ be such that

$$\begin{aligned} v(N) \leq w(S) &\leq \sum_{i \in Q_k} f_i(N, v) + |N \setminus Q_k| \alpha_{k+1} \text{ if } S = N; \\ w(S) &= v(S) \text{ if } S \subseteq Q_{k+1}; \\ |S| \beta &\leq w(S) \leq v(S) \text{ otherwise.} \end{aligned}$$

Then $f_i(N, w) = f_i(N, v)$ for all $i \in Q_k$. By core contraction independence and aggregate monotonicity, $f_i(N, w) \leq \alpha_{k+1}$ for all $i \in N \setminus Q_k$. By coalitional rationality,

$$\begin{aligned} \sum_{i \in S \setminus Q_k} f_i(N, w) &= \sum_{i \in S} f_i(N, w) - \sum_{i \in S \cap Q_k} f_i(N, w) \geq w(S) - \sum_{i \in S \cap Q_k} f_i(N, w) \\ &= v(S) - \sum_{i \in S \cap Q_k} f_i(N, v) = |S \setminus Q_k| \alpha_{k+1} \\ &= \sum_{i \in S \setminus Q_k} \alpha_{k+1} \geq \sum_{i \in S \setminus Q_k} f_i(N, w) \end{aligned}$$

for all $S \in 2^N$ with $S \not\subseteq Q_k$ for which $\frac{v(S) - \sum_{j \in S \cap Q_k} f_j(N, v)}{|S \setminus Q_k|} = \alpha_{k+1}$. This means that $f_i(N, w) = \alpha_{k+1}$ for each $i \in Q_{k+1} \setminus Q_k$. \square

Calleja, Rafels, and Tijds (2012) introduced a specific solution which satisfies coalitional rationality, aggregate monotonicity, and symmetry, but does not satisfy core contraction independence. The solution f which assigns to any game $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ for a given bijection $\sigma : \{1, \dots, |N|\} \rightarrow N$ the payoff allocation given by

$$f_{\sigma(i)}(N, v) = \max \left\{ x_{\sigma(i)} \left| x \in \prod_{k=i}^{|N|} \mathbb{R}^{\sigma(k)}, (f_{\sigma(1)}(N, v), \dots, f_{\sigma(i-1)}(N, v), x) \in C(N, v) \right. \right\}$$

for all $i \in \{1, \dots, |N|\}$, satisfies coalitional rationality, aggregate monotonicity, and core contraction independence, but does not satisfy symmetry. The coalitional Nash solution satisfies coalitional rationality, symmetry, and core contraction independence, but does not satisfy aggregate monotonicity. The equal division solution satisfies aggregate monotonicity, symmetry, and core contraction independence, but does not satisfy coalitional rationality. This means that the properties in Theorem 3.7 are independent.

Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. Recall that $\bar{x} \in \mathbb{R}^{|N|}$ is obtained from $x \in \mathbb{R}^N$ by permuting its coordinates in such a way that $\bar{x}_1 \geq \dots \geq \bar{x}_{|N|}$. A payoff allocation $y \in \mathbb{R}^{|N|}$ *lexicographically dominates* $x \in \mathbb{R}^{|N|}$, denoted by $y \prec_{\text{lex}} x$, if there exists a $k \in \{1, \dots, |N|\}$ for which $y_k < x_k$ and $y_i = x_i$ for all $i < k$. The Lmax solution (cf. Arin, Kuipers, and Vermeulen (2003)) is defined by

$$\{\text{Lmax}(N, v)\} = \{x \in C(N, v) \mid \forall y \in C(N, v) : y \not\prec_{\text{lex}} x\}.$$

In other words, the Lmax solution is the unique core element for which the maximal payoffs to the players are lexicographically minimized. Note that the Lmax solution is a specific Lorenz undominated element of the core. We show that the procedural egalitarian solution coincides with the Lmax solution on the class of egalitarian stable games.

Theorem 3.8

The procedural egalitarian solution on $\text{TU}_{\text{es}}^{\mathcal{N}}$ coincides with the Lmax solution.

Proof. Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. Denote $x = \text{PES}(N, v)$. Define $R_0 = \emptyset$ and $R_k = \{i \in N \mid \forall j \in N \setminus R_{k-1} : x_j \leq x_i\}$ for all $k \in \mathbb{N}$. Then $R_{k-1} \subseteq R_k$ for all $k \in \mathbb{N}$ and $R_k = N$ for all $k \geq |N|$. Let $y \in C(N, v)$. Let $k \in \mathbb{N}$ and assume that $y_i = x_i$ for all $i \in R_{k-1}$. Let $i \in R_k \setminus R_{k-1}$ and let $S \in \hat{\mathcal{A}}^v$ with $i \in S$ be such that $x_i \leq x_j$ for all $j \in S$. Then $S \subseteq R_k$ and

$$\sum_{j \in S \setminus R_{k-1}} y_j = \sum_{j \in S} y_j - \sum_{j \in S \cap R_{k-1}} y_j \geq v(S) - \sum_{j \in S \cap R_{k-1}} x_j = \sum_{j \in S} x_j - \sum_{j \in S \cap R_{k-1}} x_j = \sum_{j \in S \setminus R_{k-1}} x_j.$$

This means that $y_j > x_j$ for some $j \in S$ or $y_j = x_j$ for all $j \in S$. In general, $y_i > x_i$ for some $i \in R_k \setminus R_{k-1}$ or $y_i = x_i$ for all $i \in R_k \setminus R_{k-1}$. This means that there does not exist a $k \in \{1, \dots, |N|\}$ for which $\bar{y}_k < \bar{x}_k$ and $\bar{y}_i = \bar{x}_i$ for all $i < k$. Hence, the procedural egalitarian solution coincides with the Lmax solution. \square

In this section, we axiomatically characterized the procedural egalitarian solution on the class of egalitarian stable games. Besides, we showed that it unites many egalitarian concepts defined in the literature. In particular, the procedural egalitarian solution not only coincides with the Lmax solution (cf. Arin et al. (2003)) and the reduced equal split-off set *RESOS* (cf. Llerena and Mauri (2016)), but it is also an element of the strong egalitarian core *SEC* (cf. Hougaard et al. (2001)) and the egalitarian core *EC* (cf. Arin and Iñarra (2001)), which implies that it is an element of the Lorenz core *LC* (cf. Dutta and Ray (1989)) and the equal division core *EDC* (cf. Dutta and Ray (1991)). These observations are summarized in the following overview.

Corollary 3.9

Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$. Then

$$\{\text{PES}\} = \{\text{Lmax}\} = \text{RESOS} \subseteq \text{SEC} \subseteq \text{EC} \subseteq C \subseteq \text{LC} \subseteq \text{EDC} \subseteq I.^9$$

If (N, v) is convex, then

$$\{\text{PES}\} = \{\text{Lmax}\} = \text{RESOS} = \text{SEC} = \text{EC} \subseteq C \subseteq \text{LC} \subseteq \text{EDC} \subseteq I.^9$$

⁹Here, the argument (N, v) is omitted to improve readability.

4 Egalitarian stability

This section elaborates more on the domain of egalitarian stable games. Although the results of the previous section are only valid on this specific subclass of games, we believe that they are still of significant interest for the study of egalitarianism in the context of cooperative games in general, and for the trade-off between egalitarianism and coalitional rationality in particular. Considering this work as a first attempt to analyze the procedural egalitarian solution from an axiomatic point of view, we at least substantially extend the scope of the Dutta and Ray (1989) solution, which is in turn only characterized on the specific subclass of convex games. Naturally, this paper serves as a fundamental basis for an axiomatic study of the procedural egalitarian solution on the full class of transferable utility games. Nevertheless, we feel the need to relate the class of egalitarian stable games to other well-known classes of games in order to shed more light on the implications of our results. Throughout this section, (N, v) is the generic notation for a transferable utility game.

Dietzenbacher et al. (2017) showed that convex games are egalitarian stable, and that egalitarian stable games are balanced. There are many other classes of games which strictly include the class of convex games and are strictly included in the class of balanced games. One of them is the class of totally balanced games, where the core of each subgame is nonempty.

Total balancedness

$C(S, v_S) \neq \emptyset$ for all $S \in 2^N \setminus \{\emptyset\}$, where $v_S(R) = v(R)$ for each $R \subseteq S$. \circ

The game $(N, v) \in \text{TU}^N$ in Example 1 and Example 2 is egalitarian stable, but is not totally balanced since $C(\{1, 3\}, v_{\{1,3\}}) = \emptyset$. The game in Example 3 is not egalitarian stable, but is totally balanced. This means that egalitarian stability and totally balancedness are logically unrelated.

A superclass of totally balanced games is the class of games with a population monotonic allocation scheme. Inspired by Dutta and Ray (1989), Sprumont (1990) introduced this class of games to deal with the possibility of partial cooperation. A population monotonic allocation scheme (PMAS) specifies how to allocate the worth of each coalition among its members in a population monotonic way. Games for which this is possible are called PMAS admissible.

PMAS admissibility

There exists a π assigning to each $S \in 2^N \setminus \{\emptyset\}$ a payoff allocation $\pi(S) \in \mathbb{R}^S$ in such a way that $\sum_{i \in S} \pi_i(S) = v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$ and $\pi_i(S) \leq \pi_i(T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$ and each $i \in S$. \circ

The game in Example 1 and Example 2 is egalitarian stable, but is not PMAS admissible since it is not totally balanced. The game in Example 3 is not egalitarian stable, but is PMAS admissible. This is shown in the following example.

Example 7

Let $(N, v) \in \text{TU}^N$ be the game from Example 3. The worth of each coalition and the unique population monotonic allocation scheme π are presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	0	0	1	1	0	1
π	$(0, \cdot, \cdot)$	$(\cdot, 0, \cdot)$	$(\cdot, \cdot, 0)$	$(1, 0, \cdot)$	$(1, \cdot, 0)$	$(\cdot, 0, 0)$	$(1, 0, 0)$

This means that egalitarian stability and PMAS admissibility are logically unrelated. \triangle

Egalitarian stability is a *prosperity property* (cf. Van Gellekom, Potters, and Reijnierse (1999)), i.e. a game is egalitarian stable if the worth of the grand coalition is sufficiently large. Imputation admissibility and balancedness are well-known prosperity properties. Another prosperity property is largeness of the core. Sharkey (1982) introduced this property to describe games arising from economic problems involving cost allocation. A game has a large core if any vector satisfying the core inequalities dominates a core element.

Large core

For all $x \in \mathbb{R}^N$ with $\sum_{i \in S} x_i \geq v(S)$ for all $S \in 2^N$, there exists a $y \in C(N, v)$ for which $y \leq x$. \circ

Arin et al. (2003) axiomatically studied the Lmax solution on the class of large core games. Llerena and Mauri (2016) showed that the Lmax solution is the unique element of the reduced equal split-off set on the class of large core games. From Theorem 3.8 we know that the procedural egalitarian solution coincides with the Lmax solution on the class of egalitarian stable games. We prove that the class of egalitarian stable games strictly includes the class of large core games.¹⁰

Theorem 4.1

*All large core games are egalitarian stable.*¹¹

Proof. Let $(N, v) \in \text{TU}^N$ be a large core game. Since $\sum_{i \in S} \hat{\gamma}_i^v \geq v(S)$ for all $S \in 2^N$, there exists an $y \in C(N, v)$ such that $y \leq \hat{\gamma}^v$. Suppose that $y_i < \hat{\gamma}_i^v$ for some $i \in N$. Let $S \in \hat{\mathcal{A}}^v$ be such that $i \in S$. Then

$$\sum_{j \in S} y_j < \sum_{j \in S} \hat{\gamma}_j^v = v(S) \leq \sum_{j \in S} y_j.$$

This is a contradiction, so $\text{PES}(N, v) = \hat{\gamma}^v = y$. Hence, (N, v) is egalitarian stable. \square

The game $(N, v) \in \text{TU}^N$ in Example 4 is egalitarian stable, but does not have a large core since for $x = (0, 5, 0, 6)$ there does not exist a $y \in C(N, v)$ for which $y \leq x$. This means that not all egalitarian stable games are large core games.

Prosperity properties are mainly studied in relation to games with a stable core. In particular, Sharkey (1982) showed that all large core games have a stable core. A game has a stable core if any imputation outside the core is dominated by an imputation inside the core.

Stable core

$C(N, v) \neq \emptyset$ and for all $x \in I(N, v) \setminus C(N, v)$ there exist a $y \in C(N, v)$ and an $S \in 2^N \setminus \{\emptyset\}$ for which $\sum_{i \in S} y_i = v(S)$ and $y_i > x_i$ for all $i \in S$. \circ

The game $(N, v) \in \text{TU}^N$ in Example 4 is egalitarian stable, but does not have a stable core since for $x = (0, 4, 0, 4)$ there do not exist a $y \in C(N, v)$ and an $S \in 2^N \setminus \{\emptyset\}$ for which $\sum_{i \in S} y_i = v(S)$ and $y_i > x_i$ for all $i \in S$.

¹⁰Ehud Lehrer and Dries Vermeulen are gratefully acknowledged for raising this question.

¹¹In fact, on the class of large core games, all aspirations (cf. Bennett (1983)) are core elements.

Another superclass of totally balanced games is the class of exact games. A game is exact (cf. Schmeidler (1972)) if for each coalition there exists a core element which exactly distributes the corresponding worth among its members.

Exactness

For all $S \in 2^N$ there exists an $x \in C(N, v)$ for which $\sum_{i \in S} x_i = v(S)$. ○

The game in Example 1 and Example 2 is egalitarian stable, but is not exact since it is not totally balanced.

Estévez-Fernández (2012) showed that all stable core games with at most five players are large core games. Biswas, Parthasarathy, Potters, and Voorneveld (1999) showed that all exact games with at most four players are large core games. This means that all stable core games with at most five players and all exact games with at most four players are egalitarian stable. Whether all stable core games and all exact games with an arbitrary number of players are egalitarian stable is an intriguing open question. Future research could further study the class of egalitarian stable games in order to better understand the trade-off between egalitarianism and coalitional rationality.

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Appendix

Lemma A.1

The procedural egalitarian solution satisfies the equal division upper bound.

Proof. Let $(N, v) \in \text{TU}_{\text{es}}^N$. Let $i \in N$ be such that $\hat{\gamma}_i^v \geq \hat{\gamma}_j^v$ for all $j \in N$. Let $S \in \hat{\mathcal{A}}^v$ with $i \in S$ be such that $\hat{\gamma}_i^v \leq \hat{\gamma}_j^v$ for all $j \in S$. This means that $\hat{\gamma}_i^v = \frac{v(S)}{|S|}$ and

$$\text{PES}_j(N, v) = \hat{\gamma}_j^v \leq \hat{\gamma}_i^v = \frac{v(S)}{|S|} \leq \max_{T \in 2^N \setminus \{\emptyset\}} \frac{v(T)}{|T|}$$

for all $j \in N$. Hence, the procedural egalitarian solution satisfies the equal division upper bound. \square

Lemma A.2

The procedural egalitarian solution satisfies max-consistency.

Proof. Let $(N, v) \in \text{TU}_{\text{es}}^N$ and let $T \in 2^N \setminus \{\emptyset\}$. Then $\text{PES}(N, v) = \hat{\gamma}^v$. First, we show by induction that $\gamma_i^{v_T^{\text{PES}}, k} = \hat{\gamma}_i^v$ for all $k \in \mathbb{N}$ and each $i \in P^{v_T^{\text{PES}}, k}$. Let $i \in P^{v_T^{\text{PES}}, 1}$ and let $S \in \hat{\mathcal{A}}^v$ with $i \in S$ be such that $\hat{\gamma}_i^v \leq \hat{\gamma}_j^v$ for all $j \in S$. Then

$$\begin{aligned} \gamma_i^{v_T^{\text{PES}}, 1} &\geq \chi_i^{v_T^{\text{PES}}, 1}(S \cap T) = \frac{v_T^{\text{PES}}(S \cap T)}{|S \cap T|} \geq \frac{v(S) - \sum_{j \in S \setminus T} \hat{\gamma}_j^v}{|S \cap T|} \\ &= \frac{\sum_{j \in S \cap T} \hat{\gamma}_j^v}{|S \cap T|} \geq \frac{\sum_{j \in S \cap T} \hat{\gamma}_i^v}{|S \cap T|} = \frac{|S \cap T| \hat{\gamma}_i^v}{|S \cap T|} = \hat{\gamma}_i^v. \end{aligned}$$

By coalitional rationality, this means that

$$\sum_{i \in S} \gamma_i^{v_T^{\text{PES}}, 1} \geq \sum_{i \in S} \hat{\gamma}_i^v \geq \max_{R \subseteq N \setminus T} \left\{ v(S \cup R) - \sum_{i \in R} \hat{\gamma}_i^v \right\} \geq v_T^{\text{PES}}(S) = \sum_{i \in S} \gamma_i^{v_T^{\text{PES}}, 1}$$

for all $S \in \mathcal{A}^{v_T^{\text{PES}}, 1}$. This implies that $\gamma_i^{v_T^{\text{PES}}, 1} = \hat{\gamma}_i^v$ for each $i \in P^{v_T^{\text{PES}}, 1}$.

Let $k \in \mathbb{N}$ and assume that $\gamma_i^{v_T^{\text{PES}}, k} = \hat{\gamma}_i^v$ for each $i \in P^{v_T^{\text{PES}}, k}$. Let $i \in P^{v_T^{\text{PES}}, k+1} \setminus P^{v_T^{\text{PES}}, k}$ and let $S \in \hat{\mathcal{A}}^v$ with $i \in S$ be such that $\hat{\gamma}_i^v \leq \hat{\gamma}_j^v$ for all $j \in S$. Then

$$\begin{aligned} \gamma_i^{v_T^{\text{PES}}, k+1} &\geq \chi_i^{v_T^{\text{PES}}, k+1}(S \cap T) = \frac{v_T^{\text{PES}}(S \cap T) - \sum_{j \in S \cap T \cap P^{v_T^{\text{PES}}, k}} \gamma_j^{v_T^{\text{PES}}, k}}{|(S \cap T) \setminus P^{v_T^{\text{PES}}, k}|} \\ &\geq \frac{v(S) - \sum_{j \in S \setminus T} \hat{\gamma}_j^v - \sum_{j \in S \cap T \cap P^{v_T^{\text{PES}}, k}} \hat{\gamma}_j^v}{|(S \cap T) \setminus P^{v_T^{\text{PES}}, k}|} \\ &= \frac{\sum_{j \in (S \cap T) \setminus P^{v_T^{\text{PES}}, k}} \hat{\gamma}_j^v}{|(S \cap T) \setminus P^{v_T^{\text{PES}}, k}|} \\ &\geq \frac{\sum_{j \in (S \cap T) \setminus P^{v_T^{\text{PES}}, k}} \hat{\gamma}_i^v}{|(S \cap T) \setminus P^{v_T^{\text{PES}}, k}|} \\ &= \frac{|(S \cap T) \setminus P^{v_T^{\text{PES}}, k}| \hat{\gamma}_i^v}{|(S \cap T) \setminus P^{v_T^{\text{PES}}, k}|} \\ &= \hat{\gamma}_i^v. \end{aligned}$$

By coalitional rationality, this means that

$$\sum_{i \in S} \gamma_i^{v_T^{\text{PES}}, k+1} \geq \sum_{i \in S} \hat{\gamma}_i^v \geq \max_{R \subseteq N \setminus T} \left\{ v(S \cup R) - \sum_{i \in R} \hat{\gamma}_i^v \right\} \geq v_T^{\text{PES}}(S) = \sum_{i \in S} \gamma_i^{v_T^{\text{PES}}, k+1}$$

for all $S \in \mathcal{A}^{v_T^{\text{PES}}, k+1}$. This implies that $\gamma_i^{v_T^{\text{PES}}, k+1} = \hat{\gamma}_i^v$ for each $i \in P^{v_T^{\text{PES}}, k+1}$. Hence, $\hat{\gamma}^{v_T^{\text{PES}}} = \hat{\gamma}_T^v$ and

$$\sum_{i \in T} \hat{\gamma}_i^{v_T^{\text{PES}}} = \sum_{i \in T} \hat{\gamma}_i^v = v(N) - \sum_{i \in N \setminus T} \hat{\gamma}_i^v = v_T^{\text{PES}}(T).$$

This implies that $T \in \hat{\mathcal{A}}^{v_T^{\text{PES}}}$, $(T, v_T^{\text{PES}}) \in \text{TU}_{\text{es}}^{\mathcal{N}}$, and

$$\text{PES}(T, v_T^{\text{PES}}) = \hat{\gamma}^{v_T^{\text{PES}}} = \hat{\gamma}_T^v = \text{PES}_T(N, v).$$

Hence, the procedural egalitarian solution satisfies max-consistency. \square

Lemma A.3

The procedural egalitarian solution satisfies coalitional monotonicity.

Proof. Let $(N, v), (N, v') \in \text{TU}_{\text{es}}^{\mathcal{N}}$ and let $S \in 2^N$ be such that $v(S) \leq v'(S)$ and $v(T) = v'(T)$ for all $T \in 2^N \setminus \{S\}$. First, we show by induction that for all $k \in \mathbb{N}$, $\gamma_S^{v',k} \geq \hat{\gamma}_S^v$ if $S \in \mathcal{A}^{v',k}$, and $\gamma_{P^{v',k}}^{v',k} = \hat{\gamma}_{P^{v',k}}^v$ if $S \notin \mathcal{A}^{v',k}$. Let $i \in P^{v',1}$ and let $T \in \hat{\mathcal{A}}^v$ with $i \in T$ be such that $\hat{\gamma}_i^v \leq \hat{\gamma}_j^v$ for all $j \in T$. Then

$$\gamma_i^{v',1} \geq \chi_i^{v',1}(T) = \frac{v'(T)}{|T|} \geq \frac{v(T)}{|T|} \geq \hat{\gamma}_i^v.$$

This means that $\gamma_S^{v',1} \geq \hat{\gamma}_S^v$ if $S \in \mathcal{A}^{v',1}$. Suppose that $S \notin \mathcal{A}^{v',1}$. Then

$$\sum_{i \in T} \gamma_i^{v',1} \geq \sum_{i \in T} \hat{\gamma}_i^v \geq v(T) = v'(T) = \sum_{i \in T} \gamma_i^{v',1}$$

for all $T \in \mathcal{A}^{v',1}$. This means that $\gamma_i^{v',1} = \hat{\gamma}_i^v$ for each $i \in P^{v',1}$.

Let $k \in \mathbb{N}$ and assume that $\gamma_S^{v',k} \geq \hat{\gamma}_S^v$ if $S \in \mathcal{A}^{v',k}$, and $\gamma_{P^{v',k}}^{v',k} = \hat{\gamma}_{P^{v',k}}^v$ if $S \notin \mathcal{A}^{v',k}$. If $S \in \mathcal{A}^{v',k}$, then $S \in \mathcal{A}^{v',k+1}$ and $\gamma_S^{v',k+1} = \gamma_S^{v',k} \geq \hat{\gamma}_S^v$. Suppose that $S \notin \mathcal{A}^{v',k}$. Then $\gamma_{P^{v',k}}^{v',k} = \hat{\gamma}_{P^{v',k}}^v$. Let $i \in P^{v',k+1} \setminus P^{v',k}$ and let $T \in \hat{\mathcal{A}}^v$ with $i \in T$ be such that $\hat{\gamma}_i^v \leq \hat{\gamma}_j^v$ for all $j \in T$. Then

$$\gamma_i^{v',k+1} \geq \chi_i^{v',k+1}(T) = \frac{v'(T) - \sum_{j \in T \cap P^{v',k}} \gamma_j^{v',k}}{|T \setminus P^{v',k}|} \geq \frac{v(T) - \sum_{j \in T \cap P^{v',k}} \hat{\gamma}_j^v}{|T \setminus P^{v',k}|} \geq \hat{\gamma}_i^v.$$

This means that $\gamma_S^{v',k+1} \geq \hat{\gamma}_S^v$ if $S \in \mathcal{A}^{v',k+1}$. Suppose that $S \notin \mathcal{A}^{v',k+1}$. Then

$$\sum_{i \in T} \gamma_i^{v',k+1} \geq \sum_{i \in T} \hat{\gamma}_i^v \geq v(T) = v'(T) = \sum_{i \in T} \gamma_i^{v',k+1}$$

for all $T \in \mathcal{A}^{v',k+1}$. This means that $\gamma_i^{v',k+1} = \hat{\gamma}_i^v$ for each $i \in P^{v',k+1}$. Hence, $\hat{\gamma}_S^{v'} \geq \hat{\gamma}_S^v$ if $S \in \hat{\mathcal{A}}^v$, and $\hat{\gamma}_S^{v'} = \hat{\gamma}_S^v$ if $S \notin \hat{\mathcal{A}}^v$. This implies that

$$\text{PES}_S(N, v') = \hat{\gamma}_S^{v'} \geq \hat{\gamma}_S^v = \text{PES}_S(N, v).$$

Hence, the procedural egalitarian solution satisfies coalitional monotonicity. \square

Lemma A.4

The procedural egalitarian solution satisfies core contraction independence.

Proof. Let $(N, v) \in \text{TU}_{\text{es}}^{\mathcal{N}}$ and $(N, v') \in \text{TU}_{\text{es}}^{\mathcal{N}}$ be such that $\text{PES}(N, v) \in C(N, v')$ and $C(N, v') \subseteq C(N, v)$. Then $\text{PES}(N, v') \in C(N, v)$. Define $R_0 = \emptyset$ and $R_k = \{i \in N \mid \forall j \in N \setminus R_{k-1} : \hat{\gamma}_j^v \leq \hat{\gamma}_i^v\}$ for all $k \in \mathbb{N}$. Then $R_{k-1} \subseteq R_k$ for all $k \in \mathbb{N}$ and $R_k = N$ for all $k \geq |N|$. We show by induction that $\hat{\gamma}_{R_k}^{v'} = \hat{\gamma}_{R_k}^v$ for all $k \in \mathbb{N}$. Clearly, $\hat{\gamma}_i^{v'} = \hat{\gamma}_i^v$ for each $i \in R_0$.

Let $k \in \mathbb{N}$ and assume that $\hat{\gamma}_i^{v'} = \hat{\gamma}_i^v$ for each $i \in R_{k-1}$. Suppose that $\hat{\gamma}_i^{v'} > \hat{\gamma}_i^v$ for some $i \in R_k \setminus R_{k-1}$. Let $S \in \hat{\mathcal{A}}^{v'}$ with $i \in S$ be such that $\hat{\gamma}_i^{v'} \leq \hat{\gamma}_j^{v'}$ for all $j \in S$. Then

$$\begin{aligned} \sum_{j \in S} \hat{\gamma}_j^{v'} &= \sum_{j \in S \cap R_{k-1}} \hat{\gamma}_j^{v'} + \sum_{j \in S \setminus R_{k-1}} \hat{\gamma}_j^{v'} \geq \sum_{j \in S \cap R_{k-1}} \hat{\gamma}_j^v + \sum_{j \in S \setminus R_{k-1}} \hat{\gamma}_i^{v'} \\ &> \sum_{j \in S \cap R_{k-1}} \hat{\gamma}_j^v + \sum_{j \in S \setminus R_{k-1}} \hat{\gamma}_i^v \geq \sum_{j \in S \cap R_{k-1}} \hat{\gamma}_j^v + \sum_{j \in S \setminus R_{k-1}} \hat{\gamma}_j^v \\ &= \sum_{j \in S} \hat{\gamma}_j^v = \sum_{j \in S} \text{PES}_j(N, v) \geq v'(S) = \sum_{j \in S} \hat{\gamma}_j^{v'}. \end{aligned}$$

This is a contradiction, so $\hat{\gamma}_i^{v'} \leq \hat{\gamma}_i^v$ for each $i \in R_k \setminus R_{k-1}$. Let $i \in R_k \setminus R_{k-1}$ and let $S \in \hat{\mathcal{A}}^v$ with $i \in S$ be such that $\hat{\gamma}_i^v \leq \hat{\gamma}_j^v$ for all $j \in S$. Then $S \subseteq R_k$ and

$$\sum_{j \in S} \hat{\gamma}_j^v \geq \sum_{j \in S} \hat{\gamma}_j^{v'} = \sum_{j \in S} \text{PES}_j(N, v') \geq v(S) = \sum_{j \in S} \hat{\gamma}_j^v.$$

This implies that $\hat{\gamma}_i^{v'} = \hat{\gamma}_i^v$ for each $i \in R_k$. Hence, the procedural egalitarian solution satisfies core contraction independence. \square